# **I. Introduction**

We have three lectures to answer the following questions about mathematical physics equations:

* What is this?
* Why it can be interesting?
* How can we begin to analyze it?

## **1. Ordinary and partial differential equations**

* Mathematical physics equation is a partial differential equation.
* Partial differential equation is an extension of the ordinary differential equation to the case of the unknown function of many variables.
* Ordinary differential equation is an equation with unknown function under the derivative.
* This is based on the derivative notion.

### **1.1. Three interpretations of derivative**

The derivative is one of the most important concepts of mathematical analysis. It has three different interpretations.

Analytic definition: Let us consider a function *x=x*(*t*). Its derivative at a point *t*0 is the following limit if it exists



Geometric definition: Let us consider a curve. The derivative is the tangent of the angle between abscissa axe and the tangent at the considered point.



Mechanic interpretation: Average speed is defined as the ratio of the distance traveled to the travel time



Instant velocity at a time moment *t*~~0~~is the way traversed in an arbitrarily short time

### **1.2. Direct and inverse differentiation** **problems**

We can have direct and inverse differentiation problems.

Direct problems: Determine the derivative by the known function

 Determine the tangent by the known curve

 Determine the velocity by the known law of movement

Inverse problems: Determine the function by the known derivative

 Determine the curve by the known tangent

 Determine the law of movement by the known velocity

### **1.3. Differential equation and Cauchy problem**

What is the differentiation inverse problem? This is the problem of determining a function by its known derivative. We have the equation

  (1.1)

with respect to the function *x*,where the function *f* is given. This equation is called the ***differential equation***, because we have the unknown function *x* under the differentiation operation. For the easiest situation with constant *f*,we can determine the function

  (1.2)

that satisfies the equality (1.1) for all constant *c.* This equality gives the ***general solution*** of the equality (1.1) for constant *f.* Thus, the solution of the differential equation can be determined accurate to a constant. This is true for the general function *f* too. The ***partial solution*** of the considered equation is its general solution with concrete value of the constant *c*.

If we would like to determine the concrete function, then it is necessary to add an ***initial condition***

  (1.3)

where initial values of the argument *t*o and the unknown function *x*o are given. The problem (1.1), (1.3) is called the ***Cauchy problem***. Obviously, the Cauchy problem for the constant function *f* has the unique solution

  (1.4)

### **1.4. Rectilinear movement of the body**

Consider as an example the rectilinear movement of the body. By the ***second Newton law***, this movement is characterized by the equality *F = ma*, where *F ­*is the force, *m* is the mass, and *a* is the acceleration. It is known, that the acceleration is the derivative of the velocity that is the derivative of the body coordinate. Then the acceleration is the second derivative of the coordinate. Thus, the second Newton law has the form

  (1.5)

We can interpret it as the differential equation with respect to the law of movement  We have second order differential equation, because the order of the derivative of the equation here is 2. If the mass and the force are constant here, then after integration of the equality (1.5) we obtain

  (1.6)

where *c*1 is an arbitrary constant. Integrate again this equality. We get

  (1.7)

where *c*2 is an arbitrary constant. The formula (1.7) gives the general solution of the equation (1.5).

Note that the general solution of second order differential equation depends from two unknown constants. Remember that the general solution (1.2) of first order differential equation depends from one unknown constant. Of course, the general solution *n-*order differential equation depends from *n* unknown constants. If we would like to determine a concrete law of movement, it is necessary to determine to additional conditions.

By mechanics, we know that the movement equation is considered with initial conditions

  (1.8)

where *t*o is an initial time, *x*o is an initial coordinate, and *v*o is an initial velocity of the body. The equality (1.5), (1.8) are called the Cauchy problem. It has the unique solution

… (1.9)

### **1.5. Tangent plane**

Now we consider the of two variables  The function of one variable determines a curve on the plane. The function of two variables determines a surface in the space. The two-dimensional analogue of the tangent to the curve at the concrete point is the tangent plane to the surface at the concrete point. The equation of the tangent line to the curve  at the point *t*0 is determined by the formula

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The equation of the tangent plane to the surface  at the point  is determined by the formula



where  and  are the partial derivatives of the function *u* with respect to the variable *x* and *y* at the point .

The direct problem for the one-dimensional case is the calculation of the function *y* by the given function *x* (determining the tangent by the curve). The inverse for the one-dimensional case is the calculation of the function *x* by the given function *y* (determining the curve by the tangent).

The direct problem for the two-dimensional case is the calculation of the function *v* by the given function *u* (determining the tangent plane by the surface). The inverse for the one-dimensional case is the calculation of the function *u* by the given function *v* (determining the surface by the tangent plane). The inverse problem for the one-dimensional case can be transformed to the first order ***ordinary differential equation***



The inverse problem for the two-dimensional case can be transformed to the first order ***partial differential equation***

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### **1.6. First order partial differential equations**

Let us have the easy enough first order partial differential equation

 , (1.9)

where *a* is given positive constant. Consider the family of functions  with arbitrary parameter *b.* These functions are called the ***characteristics*** of the equation (1.9).

Try to determine the solution of the equation (1.9) on the characteristic with concrete parameter *b.* We have the equality

  (1.10)

Consider the function  Find its derivative. We get



because of the equality (1.10). Now we have



We have the easiest first order differential equation, which have the general solution  where *c* is an arbitrary constant. Thus, we have the equality  for all value *x.* If we would like to determine a concrete function *u = u*(*x*,*y*) it is necessary to have additional conditions.

Let us consider the equation (1.9) in the rectangle 0<*x*<*L*, 0<*y*<*M*, see Figure 1. Consider the boundary conditions

  (1.11)

How we determine the solution of the problem (1.9), (1.11) at the concrete point (*x*,*y*)? At first, it is necessary to find the characteristic that passes through this point. Determine the concrete constant *b* that satisfies the equality . We have . We can have different cases. If *b* is positive, then this characteristic passes the boundary *x=*0 of the given rectangle. The corresponding point of boundary is *η=b*. Now we determine *u*(*x*,*y*)=*f*(*η*)=*f*(*y-ax*). If *b* is negative, then this characteristic passes the boundary *y=*0 of the given rectangle. The corresponding point of boundary is ξ*=-b/a*. Now we determine *u*(*x*,*y*)=*g*(ξ)=*g*(*x-y/a*). Thus, the solution of our problem is (see Figure 1)





Figure 1. First order equation and its characteristics.

### **Conclusions**

* The function derivative has the analytic, geometric and mechanic interpretations.
* There exist the direct and inverse differentiation problems.
* The inverse differentiation problem is the determining the function by its derivative.
* The inverse differentiation problem is an ordinary differential equation.
* The ordinary differential equation is an equation with unknown function under the derivative.
* The ordinary differential equations have physical and geometric sense because of the definition of the derivative.
* The order of the equation is the maximal order of the unknown function derivative in this equation.
* The general solution of the differential equation depends from arbitrary constants.
* The quantity of the arbitrary constants is equal to the order of the equation.
* It is necessary to have additional conditions for finding partial solution of the ordinary differential equations.
* Mathematical physics equation is a partial differential equation.
* Partial differential equation is an extension of the ordinary differential equation to the case of the unknown function of many variables.
* Easiest partial differential equation is the problem of determining a surface by known tangent plane.
* This is first order partial differential equation with respect to the function of two variables.
* This equation can be solved by the characteristic method.
* It is necessary to have additional functional conditions for finding the concrete solution of the partial differential equations.

### **Next step**

We will determine that partial differential equations have not only geometric sense. There are mathematical model of many physical phenomena.

### **Task 1. Ordinary differential equations.**

It is necessary to solve the differential equation



with initial conditions



Table of parameters

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **variant** | ***t*0** | ***a*** | ***b*** | ***c*** |
| 1 | 0 | 4 | 2 | -1 |
| 2 | 1 | -4 | 1 | -2 |
| 3 | -1 | 1/4 | -1 | 2 |
| 4 | 1 | 9 | 3 | 0 |
| 5 | 2 | 1 | 2 | 2 |
| 6 | 0 | -1/4  | 0 | -1 |
| 7 | -1 | -9 | -1 | -2 |
| 8 | -2 | -1 | 1 | 4 |
| 9 | 0 | 1/9 | -1 | 0 |
| 10 | 1 | -1/9 | 1 | -1 |

**Actions**.

1. Find the general solution of the given Cauchy problem. This is  if *a=λ*2 and  if *a=-λ*2.
2. Using the initial conditions, find the constants *c*1 and *c*2.
3. Put these constants to the formula of the general solution. The result is the solution of the problem.
4. Make sure that the result satisfies the given equation and initial conditions.

### **Добавки в лекцию**

1. Случай главной характеристики и возможность разрывных решений
2. Где можно, а где нельзя задавать граничные условия
3. Случай убывающих характеристик